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The modulation of nonlinear dispersive waves in a warm plasma stream

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Abstract. The weakly nonlinear dispersive modulation of one-dimensional waves in a warm, collisionless, field-free, slightly non-uniform, streaming electron plasma is investigated. Equations governing the coupled slow modulations of the waves and the slow variation of the background plasma are obtained using a two-timing procedure devised by Gatignol. Under some restrictive assumptions the complex wave amplitude is shown to vary according to a nonlinear Schrödinger equation.

1. Introduction

Several authors have described methods for investigating nonlinear dispersive wave modulation when the wave amplitude is small, and dispersive and nonlinear effects are comparable. For example Kakutani and Sugimoto (1974), using an extension of the Krylov–Bogoliubov–Mitropolsky perturbation method applied to a model equation, showed that the complex wave amplitude was slowly varying according to a nonlinear Schrödinger equation. Kawahara (1973) obtained the same result using the 'derivative expansion' method. In these papers it was assumed that, to first order in a small parameter characterising the wave amplitude, the mean values of the perturbed quantities in the wave were just their equilibrium values, and that to zero order the frequency and wavenumber of the waves were not modulated. These restrictive hypotheses were not assumed by Gatignol (1977), who described a two-timing procedure which gave a more general system of modulation equations. By introducing the restrictions he was able to recover the nonlinear Schrödinger equation as expected.

Kakutani and Sugimoto (1974) applied their method to several plasma systems including a warm electron plasma, while Gatignol (1977) illustrated his method by applying it to ion-acoustic waves. In both these papers it was assumed that the parameters describing the background plasma were constant and so it was convenient to write the basic equations in non-dimensional form. It was also assumed that the unperturbed plasma was at rest in the laboratory frame.

The purpose of this paper is to apply Gatignol's method to waves in a streaming warm electron plasma in which the background plasma is allowed to vary slowly. In § 2 the basic equations are given and the application of Gatignol's method is described. A system of equations is derived, including the equation for conservation of wave action and the dispersion relation, which govern the coupled slow modulations of the wave and the slow variation of the background plasma. In § 3 the restrictive assumptions referred to above are imposed and, for the case where the background plasma is uniform, the nonlinear Schrödinger equation governing the slow variation of the complex amplitude is derived.

This paper complements the work of Gribben and Parkes (1981), where the same plasma system is studied using an averaging method and under the assumption that nonlinear effects dominate dispersive effects.

2. Application of Gatignol's method

With the assumptions and notation used in Gribben and Parkes (1981) the basic equations for the problem can be written as

$$\partial n/\partial t+\partial(nv)/\partial x=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}\left(v^2+\frac{3cn^2}{m}\right)+\frac{eE}{m}=0, \tag{2}$$

$$\partial E/\partial x+(e/\varepsilon_0)(n-N)=0, \tag{3}$$

where E is the electric field. The equilibrium state is $n=N$, $v=U$, $E=0$, where U is the streaming velocity of the plasma. The exact harmonic solution of the linearised system of basic equations is

$$n=N+\varepsilon(\varepsilon_0 kA/e)\sin\theta, \qquad v=U+\varepsilon[\varepsilon_0(\omega-kU)A/eN]\sin\theta, \qquad E=\varepsilon A\cos\theta,$$

where ε is a small parameter characterising the wave amplitude. The phase variable θ is defined by $\theta=kx-\omega t$, and the frequency ω and wavenumber k satisfy the linear dispersion relation

$$-(\omega-kU)^2+\omega_{\rm p}^2+(kc_{\rm s})^2=0, \tag{4}$$

where $\omega_{\rm p}^2=Ne^2/\varepsilon_0 m$ and $c_{\rm s}^2=3cN^2/m$.

To study slowly modulated trains of waves of the above form, in which nonlinear and dispersive effects are taken into account and are comparable, we appeal to the reasoning of Gatignol (1977) and look for the unknowns in the form

$$\begin{aligned} n&=N_0+\varepsilon(N_1,\cos\theta+N_1'\sin\theta)+\varepsilon^2(N_2\cos 2\theta+N_2'\sin 2\theta)+{\rm O}(\varepsilon^3),\\ v&=U_0+\varepsilon(U_1\cos\theta+U_1'\sin\theta)+\varepsilon^2(U_2\cos 2\theta+U_2'\sin 2\theta)+{\rm O}(\varepsilon^3),\\ E&=\varepsilon(E_0+E_1\cos\theta+E_1'\sin\theta)+\varepsilon^2(E_2\cos 2\theta+E_2'\sin 2\theta)+{\rm O}(\varepsilon^3),\end{aligned} \tag{5}$$

where the definition of θ is now generalised to

$$\partial\theta/\partial t=-\omega, \qquad \partial\theta/\partial x=k. \tag{6}$$

The quantities ω, k, N_0, N_1, etc are functions of ε and the independent variables $X=\varepsilon x$, $T=\varepsilon t$ which correspond to variations on the slow scale. From (6) it follows that

$$\partial k/\partial T+\partial\omega/\partial X=0. \tag{7}$$

We shall use the notation $\{\ \}^{\rm I}$, $\{\ \}^{\rm II}$ to denote the approximate value of the quantity within the braces up to first and second order in ε respectively, while $\{\ \}^0$, $\{\ \}^1$, $\{\ \}^2$ will denote the zeroth-, first- and second-order parts of that quantity respectively.

Hence, for example, $\{\omega\}^{\mathrm{II}}=\{\omega\}^0+\{\omega\}^1+\{\omega\}^2$. Clearly $\{N_0\}^0=N$ and $\{U_0\}^0=U$. Also, without loss of generality, we can choose θ so that $E_1'=0$ to all orders.

The expansions (5) are substituted into the basic equations (1)–(3), and in each of these equations we equate to zero the θ-independent term and the coefficients of $\cos\theta$, $\sin\theta$ to second order, and the coefficients of $\cos 2\theta$, $\sin 2\theta$ to zero order, respectively. We obtain the following systems of equations.

(i) From the θ-independent terms in (1), (2), (3) we obtain the system

$$\left\{\frac{\partial N_0}{\partial T}+\frac{\partial}{\partial X}(N_0U_0)\right\}^{\mathrm{II}}+\frac{\varepsilon^2}{2}\left\{\frac{\partial}{\partial X}(N_1,U_1+N_1'U_1')\right\}^0=0,$$

$$\left\{\frac{\partial U_0}{\partial T}+\frac{1}{2}\frac{\partial}{\partial X}\left(U_0^2+\frac{3c}{m}N_0^2\right)+\frac{eE_0}{m}\right\}^{\mathrm{II}}+\frac{\varepsilon^2}{4}\left\{\frac{\partial}{\partial X}\left((U_1^2+U_1'^2)+\frac{3c}{m}(N_1^2+N_1'^2)\right)\right\}^0=0,$$

$$(e/\varepsilon_0)\{N_0-N\}^{\mathrm{II}}+\varepsilon^2\{\partial E_0/\partial X\}^0=0. \tag{8}$$

(ii) The $\cos\theta$ and $\sin\theta$ coefficient systems split, at each order, into two independent systems. One for the unknowns N_1, U_1 is obtained from the $\sin\theta$ coefficients from (1), (2) and the $\cos\theta$ coefficient from (3):

$$\begin{aligned}&\{\tilde{\omega}N_1-kN_0U_1\}^{\mathrm{II}}+\varepsilon\left\{\frac{\partial N_1'}{\partial T}+\frac{\partial}{\partial X}(U_0N_1'+N_0U_1')\right\}^{\mathrm{I}}\\&\qquad-\tfrac{1}{2}\varepsilon^2\{k(N_1U_2+N_1'U_2'+N_2U_1+N_2'U_1')\}^0=0,\\&\left\{\tilde{\omega}U_1-\frac{3c}{m}kN_0N_1\right\}^{\mathrm{II}}+\varepsilon\left\{\frac{\partial U_1'}{\partial T}+\frac{\partial}{\partial X}\left(U_0U_1'+\frac{3c}{m}N_0N_1'\right)\right\}^{\mathrm{I}}\\&\qquad-\tfrac{1}{2}\varepsilon^2\{k[(U_1U_2+U_1'U_2')+(3c/m)(N_1N_2+N_1'N_2')]\}^0=0,\\&(e/\varepsilon_0)\{N_1\}^{\mathrm{II}}+\varepsilon\{\partial E_1/\partial X\}^{\mathrm{I}}=0,\end{aligned}\tag{9}$$

where $\tilde{\omega}=\omega-kU_0$, and a second system for the unknowns N_1', U_1', E_1 is obtained from the $\cos\theta$ coefficients from (1), (2) and the $\sin\theta$ coefficient from (3):

$$\begin{aligned}&\{kN_0U_1'-\tilde{\omega}N_1'\}^{\mathrm{II}}+\varepsilon\left\{\frac{\partial N_1}{\partial T}+\frac{\partial}{\partial X}(N_1U_0+N_0U_1)\right\}^{\mathrm{I}}\\&\qquad+\tfrac{1}{2}\varepsilon^2\{k(N_1U_2'-U_2N_1'-N_2U_1'+U_1N_2')\}^0=0,\\&\left\{\frac{3c}{m}N_0N_1'-\tilde{\omega}U_1'+\frac{eE_1}{m}\right\}^{\mathrm{II}}+\varepsilon\left\{\frac{\partial U_1}{\partial T}+\frac{\partial}{\partial X}\left(U_0U_1+\frac{3c}{m}N_0N_1\right)\right\}^{\mathrm{I}}\\&\qquad+\tfrac{1}{2}\varepsilon^2\{k[(U_1U_2'-U_2U_1')+(3c/m)(N_1N_2'-N_2N_1')]\}^0=0,\\&\{-kE_1,+(e/\varepsilon_0)N_1'\}^{\mathrm{II}}=0.\end{aligned}\tag{10}$$

System (9), with three equations for two unknowns, leads to a compatibility condition identified as the conservation of wave action equation. System (10) is homogeneous at each order and provides the dispersion relation.

(iii) In a similar way the $\sin 2\theta$ coefficients from (1), (2) and the $\cos 2\theta$ coefficient from (3) give a system for the unknowns N_2, U_2, E_2':

$$\{2\tilde{\omega}N_2-k(2N_0U_2+N_1U_1-N_1'U_1')\}^0=0,$$

$$\{4\tilde{\omega}U_2-k[U_1^2-U_1'^2+(3c/m)(4N_0N_2+N_1^2-N_1'^2)]+2eE_2'/m\}^0=0,$$

$$\{2kE_2'+(e/\varepsilon_0)N_2\}^0=0,$$

and the $\cos 2\theta$ coefficients from (1), (2) and the $\sin 2\theta$ coefficient from (3) give a system for the unknowns N_2', U_2', E_2.

$$\{2\tilde{\omega}N_2' - k(2N_0U_2' + N_1U_1' + U_1N_1')\}^0 = 0,$$
$$\{2\tilde{\omega}U_2' - k[U_1U_1' + (3c/m)(2N_0N_2' + N_1N_1')] - eE_2/m\}^0 = 0,$$
$$\{-2kE_2 + (e/\varepsilon_0)N_2'\}^0 = 0.$$

As the steps involved in obtaining the solution of the above systems of equations are similar to those described by Gatignol (1977) for the ion-acoustic wave, they are not detailed here. We merely quote the final results.

There are sixteen unknowns in all, of which ten are as follows:

$$\{N_1\}^{\mathrm{II}} = -\frac{\varepsilon\varepsilon_0}{e}\left\{\frac{\partial E_1}{\partial X}\right\}^{\mathrm{I}}, \qquad \{U_1\}^{\mathrm{II}} = \frac{\varepsilon\varepsilon_0}{eN}\left\{\frac{\partial E_1}{\partial T} + U_0\frac{\partial E_1}{\partial X}\right\}^{\mathrm{I}}, \qquad \{N_1'\} = \frac{\varepsilon_0}{e}\{kE_1\}^{\mathrm{II}},$$

$$\{U_1'\}^{\mathrm{II}} = \frac{\varepsilon_0}{eN}\{\tilde{\omega}E_1\}^{\mathrm{II}} + \frac{\varepsilon^2 e\varepsilon_0}{mN\omega_{\mathrm{p}}^2}\left\{\tilde{\omega}E_1\left(\frac{\partial E_0}{\partial X} - \frac{(kE_1)^2}{12m\omega_{\mathrm{p}}^4}(16k^2c_{\mathrm{s}}^2 + 9\omega_{\mathrm{p}}^2)\right)\right\}^0,$$

$$\{N_2\}^0 = \frac{-\varepsilon_0}{3m\omega_{\mathrm{p}}^4}\{(4k^2c_{\mathrm{s}}^2 + 3\omega_{\mathrm{p}}^2)(kE_1)^2\}^0, \qquad \{N_2'\}^0 = 0,$$

$$\{U_2\}^0 = \frac{-\varepsilon_0}{6mN\omega_{\mathrm{p}}^4}\{(8k^2c_{\mathrm{s}}^2 + 3\omega_{\mathrm{p}}^2)\tilde{\omega}kE_1^2\}^0, \qquad \{U_2'\}^0 = 0,$$

$$\{E_2'\}^0 = \frac{e}{6m\omega_{\mathrm{p}}^4}\{(4k^2c_{\mathrm{s}}^2 + 3\omega_{\mathrm{p}}^2)kE_1^2\}^0, \qquad \{E_2\}^0 = 0.$$

The coupled slow variations of the remaining six unknowns, $\{\omega\}^{\mathrm{II}}$, $\{k\}^{\mathrm{II}}$, $\{N_0\}^{\mathrm{II}}$, $\{U_0\}^{\mathrm{II}}$, $\{E_0\}^{\mathrm{II}}$, $\{E_1\}^{\mathrm{I}}$ are given by (11)–(16)†.

$$\left\{\frac{\partial N_0}{\partial T} + \frac{\partial}{\partial X}(N_0U_0)\right\}^{\mathrm{II}} + \frac{\varepsilon^2\varepsilon_0}{2m}\left\{\frac{\partial}{\partial X}\left(\frac{\tilde{\omega}kE_1^2}{\omega_{\mathrm{p}}^2}\right)\right\}^0 = 0, \tag{11}$$

$$\left\{\frac{\partial U_0}{\partial T} + \frac{1}{2}\frac{\partial}{\partial X}\left(U_0^2 + \frac{3c}{m}N_0^2\right) + \frac{eE_0}{m}\right\}^{\mathrm{II}} + \frac{\varepsilon^2\varepsilon_0}{4m}\left\{\frac{\partial}{\partial X}\left((\tilde{\omega}^2 + k^2c_{\mathrm{s}}^2)\frac{E_1^2}{N\omega_{\mathrm{p}}^2}\right)\right\}^0 = 0, \tag{12}$$

$$\{N_0\}^{\mathrm{II}} = N - (\varepsilon^2\varepsilon_0/e)\{\partial E_0/\partial X\}^0, \tag{13}$$

$$\left\{\frac{\partial}{\partial T}\left(\frac{\tilde{\omega}E_1^2}{N}\right) + \frac{\partial}{\partial X}\left(\frac{V_{\mathrm{G}}\tilde{\omega}E_1^2}{N}\right)\right\}^{\mathrm{I}} = 0, \tag{14}$$

$$\{\mathscr{D}\}^{\mathrm{II}} + (\varepsilon^2\varepsilon_0/6mN\omega_{\mathrm{p}}^4)\{[16(kc_{\mathrm{s}})^4 + 21(\omega_{\mathrm{p}}kc_{\mathrm{s}})^2 + 6\omega_{\mathrm{p}}^4](kE_1)^2\}^0$$
$$-\varepsilon^2\left\{\frac{N}{E_1}\left[\frac{\partial}{\partial X}\left(\frac{3cN}{m}\frac{\partial E_1}{\partial X}\right) - N\frac{\partial}{\partial \eta}\left(\frac{1}{N^2}\frac{\partial E_1}{\partial \eta}\right)\right]\right\}^0 = 0, \tag{15}$$

$$\{\partial k/\partial T + \partial\omega/\partial X\}^{\mathrm{II}} = 0, \tag{16}$$

† When $\{N_1'\}^{\mathrm{II}}$, $\{U_1'\}^{\mathrm{II}}$, which involve $\{E_1\}^2$, are eliminated from system (10) to obtain the dispersion relation, it is found that the coefficient of $\{E_1\}^2$ is $\{\mathscr{D}\}^0$, i.e. zero. Hence E_1 only appears up to first order in (11)–(16).

where

$$\mathscr{D}(\tilde{\omega}, k) \equiv -\tilde{\omega}^2 + (3c/m)N_0^2k^2 + N_0e^2/\varepsilon_0 m,$$
$$\{V_G\}^{\mathrm{I}} = \{U_0 + 3ckN_0^2/m\tilde{\omega}\}^{\mathrm{I}}, \tag{17}$$
$$\partial/\partial\eta \equiv \partial/\partial T + U\,\partial/\partial X.$$

Equations (11)–(13) come from system (8), equation (14) is the equation for conservation of wave action, (15) is the dispersion relation and (16) comes from (7). V_G is the nonlinear group velocity defined by $V_G = \partial\omega/\partial k$.

To zero order (15) gives $\{\mathscr{D}\}^0 = 0$ which is just the linear dispersion relation (4). To first order $\{\mathscr{D}\}^{\mathrm{I}} = 0$ and this is used to obtain (17). The second-order dispersion relation (15) has a linear term involving derivatives of $\{E_1\}^0$ (the zero-order wave amplitude) due to dispersive effects and a term involving $\{E_1^2\}^0$ due to nonlinear effects. In Gribben and Parkes (1981), where dispersive effects were assumed to be of a smaller order than the nonlinear effects, the second-order dispersion relation did not involve derivatives. Note that for ion-acoustic waves, Gatignol (1977) showed that the dispersive term also includes derivatives of ω and k.

3. Derivation of the nonlinear Schrödinger equation

We now assume that to first order the mean values of the perturbed quantities in the wave are just their equilibrium values. From (13) $\{N_0\}^1$ is zero already, but now we assume that $\{U_0\}^1 = 0$ and $\{E_0\}^0 = 0$ as well. From (13) the latter assumption implies that $\{N_0\}^2 = 0$. If we also assume that to zero order ω and k are not modulated, then we can write

$$\omega = \omega_0 - \varepsilon\,\partial\alpha/\partial T, \qquad k = k_0 + \varepsilon\,\partial\alpha/\partial X,$$

so that (16) is automatically satisfied, where ω_0, k_0 are constants and α is a function of X, T and ε. It follows that if we set $\{U_0\}^2 = \varepsilon^2\tilde{U}$ then

$$\{\tilde{\omega}\}^{\mathrm{II}} = \{\tilde{\omega}_0 - \varepsilon(\partial\alpha/\partial T + U\partial\alpha/\partial X) - \varepsilon^2 k_0\tilde{U}\}^{\mathrm{II}}, \tag{18}$$

where $\tilde{\omega}_0 \equiv \omega_0 - k_0 U$.

To zero order (11) and (12) give

$$\frac{\partial N}{\partial T} + \frac{\partial}{\partial X}(NU) = 0, \qquad \frac{\partial U}{\partial T} + \frac{1}{2}\frac{\partial}{\partial X}\left(U^2 + \frac{3c}{m}N^2\right) = 0, \tag{19}$$

and the second-order part of (11) gives

$$\frac{\partial}{\partial X}(N\tilde{U}) + \frac{\partial}{\partial X}\left\{\frac{\varepsilon_0 k_0\tilde{\omega}_0 E_1^2}{2m\omega_p^2}\right\}^0 = 0,$$

whence

$$\tilde{U} = -\{\varepsilon_0 k_0\tilde{\omega}_0 E_1^2/2mN\omega_p^2\}^0 + \gamma/N;$$

where $\gamma = \gamma(T)$. Henceforth we shall consider the case where N, U are both constant, which is a possible solution of (19). This implies that $\tilde{\omega}_0$ is also constant. For convenience we non-dimensionalise X, T, $\tilde{\omega}$, k, E_1, V_G by c_s/ω_p, ω_p^{-1}, ω_p, ω_p/c_s, $m\omega_p c_s/e$, c_s, respectively.

The equation for conservation of wave action, (14), and the dispersion relation, (15), can be written

$$\left\{\frac{\partial E_1}{\partial T}+V_g\frac{\partial E_1}{\partial X}\right\}^{\mathrm{I}}+\frac{\varepsilon}{\tilde{\omega}_0^3}\left\{\frac{\partial E_1}{\partial X}\frac{\partial \alpha}{\partial X}+\frac{E_1}{2}\frac{\partial^2\alpha}{\partial X^2}\right\}^0=0, \tag{20}$$

$$\left\{\left(\frac{\partial \alpha}{\partial T}+V_g\frac{\partial \alpha}{\partial X}\right)E_1\right\}^{\mathrm{I}}+\varepsilon\left\{\frac{E_1}{2\tilde{\omega}_0^3}\left(\frac{\partial\alpha}{\partial X}\right)^2+\frac{E_1^3k_0^4}{12\tilde{\omega}_0}(16k_0^2+15)+E_1\gamma k_0-\frac{1}{2\tilde{\omega}_0^3}\frac{\partial^2E_1}{\partial X^2}\right\}^0=0, \tag{21}$$

where $V_g\equiv\{V_G\}^0=U+k_0/\tilde{\omega}_0$. The corresponding zero-order results

$$\left\{\frac{\partial E_1}{\partial T}+V_g\frac{\partial E_1}{\partial X}\right\}^0=0 \qquad \text{and} \qquad \left\{\frac{\partial \alpha}{\partial T}+V_g\frac{\partial \alpha}{\partial X}\right\}^0=0,$$

the zero-order dispersion relation $\{\mathscr{D}\}^0=0$, and (18) have been used to effect some simplification. To conform with the notation of Kakutani and Sugimoto (1974), we introduce a complex amplitude a by $a=\frac{1}{2}E_1e^{i\alpha}$ and form the combination [i(20)−(21)]$e^{i\alpha}$ to obtain

$$i\{\partial a/\partial T+V_g\,\partial a/\partial X\}^{\mathrm{I}}+\varepsilon P\,\partial^2a/\partial X^2=\varepsilon\{Q|a|^2a+Ra\}^0,$$

where

$$P=1/2\tilde{\omega}_0^3, \qquad Q=k_0^4(16k_0^2+15)/3\tilde{\omega}_0, \qquad R=\gamma k_0.$$

This can be transformed into the nonlinear Schrödinger equation

$$i\,\partial a/\partial\tau+P\,\partial^2a/\partial\xi^2=Q|a|^2a \tag{22}$$

by introducing the coordinate transformation $\xi=X-V_gT$, $\tau=\varepsilon T$ and the phase shift $a\to a\exp[-i\int^\tau R(\tau')\,d\tau']$. Equation (22), with the streaming velocity U set to zero, agrees with the result of Kakutani and Sugimoto (1974).

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